PROOF OF THE ISOPERIMETRIC INEQUALITY

Our goal is to show that balls satisfy the isoperimetric inequality. Let us fix some notation. We are given a number $m > 0$ and we aim to solve the isoperimetric problem, that is,

$$\inf \Big\{ P(E),\, E \subseteq \mathbb{R}^N,\, |E| = m \Big\} ,$$

where the infimum is taken among all the sets of finite perimeter. We will call $\mathfrak{J}(m)$ the value of the above infimum, and we will show the following result.

Theorem 1. *The infimum above is reached by a set E if and only if E is a ball of volume m.*

Notice that, as an immediate consequence of this theorem, we will obtain

$$\mathfrak{J}(m) = N\omega_N^{1/N} m^{\frac{N-1}{N}} .$$

We can start by finding a weaker version of this formula.

Lemma 2. *For any $m > 0$ one has*

$$\mathfrak{J}(m) = m^{\frac{N-1}{N}}\, \mathfrak{J}(1) .$$

Proof. This is obvious by rescaling, since for every set E of finite perimeter and for every $\lambda > 0$ we have

$$|\lambda E| = \lambda^N |E| , \qquad\qquad P(\lambda E) = \lambda^{N-1} P(E) .$$

□

Thanks to this simple formula, we can immediately show the existence of isoperimetric sets.

Lemma 3. *There exist isoperimetric sets of every volume.*

Proof. Let $\{E_n\}_{n\in\mathbb{N}}$ be an isoperimetric sequence, that is, a sequence such that

$$|E_n| = m , \qquad\qquad P(E_n) \xrightarrow[n\to\infty]{} \mathfrak{J}(m) .$$

The corresponding sequence χ_{E_n} is then bounded in $BV(\mathbb{R}^N)$, so up to a subsequence we have that $\chi_{E_n} \stackrel{*}{\rightharpoonup} f$ in $BV_{\rm loc}$ for some function $f \in BV_{\rm loc}(\mathbb{R}^N)$. Since the convergence is strong in $L^1_{\rm loc}(\mathbb{R}^N)$, we deduce that f admits only 0 and 1 as values, so actually $f = \chi_{\overline{E}}$ for some set $\overline{E} \subseteq \mathbb{R}^N$. By lower semicontinuity, we obtain

$$|\overline{E}| = \|\chi_{\overline{E}}\|_{L^1} \leq \liminf |E_n| = m , \qquad P(\overline{E}) = |D\chi_{\overline{E}}|(\mathbb{R}^N) \leq \liminf P(E_n) = \mathfrak{J}(m) .$$

As a consequence, we will conclude the proof, being the set $\overline{E}$ is an isoperimetric set of volume m, as soon as we will find an isoperimetric sequence $\{E_n\}$ such that the corresponding limit set $\overline{E}$ satisfies

$$|\overline{E}| = m . \tag{1}$$

To obtain such a sequence, let us fix $\varepsilon > 0$ and consider a generic set E with volume m and perimeter $P(E) < \mathfrak{J}(m) + \varepsilon$. Let moreover $t \in \mathbb{R}$ be a chosen so that $\mathscr{H}^{N-1}(E \cap \{x_1 = t\}) < \varepsilon$. Let us also set $E^- = E \cap \{x_1 < t\}$ and $E^+ = E \cap \{x_1 > t\}$, and $m_1 = |E^-|$, $m_2 = |E^+| = m - m_1$. By Lemma 2 one can evaluate

$$\begin{aligned} m^{\frac{N-1}{N}} \mathfrak{J}(1) + \varepsilon = \mathfrak{J}(m) + \varepsilon > P(E) &\geq P(E^-) + P(E^+) - 2\mathscr{H}^{N-1}(E \cap \{x_1 = t\}) \\ &> P(E^-) + P(E^+) - 2\varepsilon \geq \mathfrak{J}(m_1) + \mathfrak{J}(m_2) - 2\varepsilon = J(1)\Big(m_1^{\frac{N-1}{N}} + m_2^{\frac{N-1}{N}}\Big) - 2\varepsilon\,. \end{aligned}$$

Since $t \mapsto t^{\frac{N-1}{N}} + (m-t)^{\frac{N-1}{N}}$ is a strictly concave map with minimum exactly at $t = 0$ and $t = m$, we deduce that

$$\min\{m_1,\, m_2\} \leq \delta(\varepsilon)$$

for some continuous and increasing function $\varepsilon \mapsto \delta(\varepsilon)$ satisfying $\delta(0) = 0$ (it is possible to write explicitely the function $\delta(\varepsilon)$, but there is no need to do so). Up to a translation, we can assume that $|E \cap \{x_1 > 0\}| = m/2$. By Fubini Theorem, it is possible to find $0 < t < m/\varepsilon$ satisfying $\mathscr{H}^{N-1}(E \cap \{x_1 = t\}) < \varepsilon$. The above argument implies that

$$\big|E \cap \{x_1 \geq m/\varepsilon\}\big| \leq \big|E \cap \{x_1 \geq t\}\big| \leq \delta(\varepsilon)\,.$$

The symmetric argument implies that also $|E \cap \{x_1 < -m/\varepsilon\}| \leq \delta(\varepsilon)$, and of course the very same can be done for every direction. Summarizing, if a set E of volume m satisfies $P(E) < \mathfrak{J}(m) + \varepsilon$, then up to a translation we have

$$\Big|E \cap [-M, M]^N\Big| > m - 2N\delta(\varepsilon)\,,$$

calling $M = m/\varepsilon$.

Consider then any isoperimetric sequence $\{E_n\}$ of volume m. Up to translate the sets, we can assume that $|E_n \cap \{x_j > 0\}| = m/2$ for every $n \in \mathbb{N}$ and every $1 \leq j \leq N$. Up to a subsequence, we know that $\chi_{E_n} \stackrel{*}{\rightharpoonup} \chi_{\overline{E}}$ for some set $\overline{E}$. For every ε, the inequality $P(E_n) < \mathfrak{J}(m) + \varepsilon$ is true for each n big enough. So, recalling that the convergence of χ_{E_n} to χ_E is strong in $L^1([-M, M]^N)$, we deduce

$$|\overline{E}| \geq \Big|\overline{E} \cap [-M, M]^N\Big| \geq \lim_{n\to\infty} \Big|E_n \cap [-M, M]^N\Big| \geq m - 2N\delta(\varepsilon)\,,$$

and since $\varepsilon > 0$ is arbitrary we obtain the validity of (1). The proof is then concluded. □

Up to now, we only have the existence of isoperimetric sets. We must show that these sets are exactly the balls. We will strongly use the symmetry of the problem. Our first tool, very simple yet fundamental, is the following symmetrization argument.

Lemma 4. *Let E be a set of finite perimeter, and let Π be a $(N-1)$-dimensional affine hyperplane bisecting E, that is, the volume of E in the two half-spaces corresponding to the plane Π is the same. Let us call E_1 and E_2 the two sets symmetric with respect to Π and which coincide with E in one of the two half-spaces. Then,*

$$P(E) \geq \frac{P(E_1) + P(E_2)}{2}\,. \tag{2}$$

Proof. Let us call $\pi : \mathbb{R}^N \to \mathbb{R}^N$ the reflection with respect to Π, let us call $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$ the two open half-spaces induced by Π, and let $E^- = E \cap \mathbb{R}^N_-$ and $E^+ = E \cap \mathbb{R}^N_+$. Notice that $E_1 = E^- \cup \pi(E^-)$ and $E_2 = E^+ \cup \pi(E^+)$. We can observe that

$$\mathscr{H}^{N-1}\Big(\partial^* E_1 \setminus \big((\partial^* E \cap \mathbb{R}^N_-) \cup \pi(\partial^* E \cap \mathbb{R}^N_-) \cup \partial^* E\big)\Big) = 0\,. \tag{3}$$

(As we will describe in Remark 6 something stronger can be said, but this estimate is enough for our purposes). To show (3), we first observe that $\partial^* E_1 \cap \mathbb{R}^N_- = \partial^* E \cap \mathbb{R}^N_-$ and $\partial^* E_1 \cap \mathbb{R}^N_+ = \pi(\partial^* E \cap \mathbb{R}^N_-)$, so we only have to check that $\mathscr{H}^{N-1}\big(\partial^* E_1 \cap (\Pi \setminus \partial^* E)\big) = 0$. But in fact, by the results of the Appendix, we know that $\mathscr{H}^{N-1}$-a.e. point of $\Pi \setminus \partial^* E$ has density with respect to E either 0, or 1, and in both cases its density with respect to E_1 is again 0 or 1, and not 1/2, so (3) is established. We deduce

$$P(E_1) = \mathscr{H}^{N-1}(\partial^* E_1) \le 2\mathscr{H}^{N-1}(\partial^* E \cap \mathbb{R}^N_-) + \mathscr{H}^{N-1}(\partial^* E \cap \Pi)\,.$$

The very same estimate, with E_2 in place of E_1, gives then

$$P(E_2) \le 2\mathscr{H}^{N-1}(\partial^* E \cap \mathbb{R}^N_+) + \mathscr{H}^{N-1}(\partial^* E \cap \Pi)\,,$$

and adding up these two estimates we get (2). □

Corollary 5. *There exist an isoperimetric set of volume m which is symmetric with respect to a hyperplane. There also exists an isoperimetric set which is symmetric with respect to N orthogonal hyperplanes.*

Proof. We know that an isoperimetric set E exists by Lemma 3. It is clearly possible to find an affine hyperplane bisecting E with any given normal direction (it is enough to translate the hyperplane until it bisects the set). Calling E_1 and E_2 the symmetric sets defined in Lemma 4, we have then the validity of (2). Since both E_1 and E_2 have the same volume as E by construction, by minimality of E we deduce that both E_1 and E_2 are isoperimetric sets.

To obtain an isoperimetric set symmetric with respect to N orthogonal hyperplanes, it is enough to argue by induction. Indeed, take an isoperimetric set E symmetric with respect to j orthogonal hyperplanes, with $1 \le j < N$. Then, let Π be a hyperplane bisecting E and orthogonal to all the j existing hyperplanes. As noticed before, calling E_1 and E_2 the sets obtaining by symmetrization of E with respect to E, they are both isoperimetric sets of volume m. And finally, since the hyperplanes are orthogonal, not only E_1 and E_2 are symmetric with respect to Π, but they are symmetric also with respect to the previous j hyperplanes, so they are both symmetric with respect to $j+1$ orthogonal hyperplanes. This concludes the thesis. □

Remark 6. *It is simple to observe that an estimate more precise than (3) actually holds, i.e.,*

$$\partial^* E_1 = (\partial^* E \cap \mathbb{R}^N_-) \cup \pi(\partial^* E \cap \mathbb{R}^N_-) \qquad \mathscr{H}^{N-1} - a.e.\,.$$

In other words, $\mathscr{H}^{N-1}(\partial^* E_1 \cap \Pi) = 0$. Indeed, while proving (3) we have observed that $\partial^* E_1 \cap \Pi \subseteq \partial^* E \cap \Pi$ up to $\mathscr{H}^{N-1}$-negligible subsets. But a point $x \in \Pi \cap \partial^* E$ can belong to $\partial^* E_1$ only if the normal vector to E at x lies inside Π. On the other hand, it is not difficult to observe that $\mathscr{H}^{N-1}$-a.e. point of $\partial^* E \cap \Pi$ has normal vector orthogonal to Π.

We can now prove a much more involved estimate, by making use of the Steiner symmetrization, which we now introduce.

Definition 7 (Steiner symmetrization)**.** *Let $E \subseteq \mathbb{R}^N$ be a set of finite volume. For $\mathscr{H}^{N-1}$a.e. $x' \in \mathbb{R}^{N-1}$ the function $\varphi(x') = \mathscr{H}^1(E_{x'})/2$ is well defined, where $E_{x'} \subseteq \mathbb{R}$ is the* section of E at x' *given by $E_{x'} = \{t \in \mathbb{R} : (x', t) \in E\}$.*

The Steiner symmetrized of E *is then the set*

$$E^* = \Big\{(x', t) \in \mathbb{R}^N : -\varphi(x') < t < \varphi(x')\Big\}.$$

Notice that E^* is symmetric with respect to the hyperplane $\{x_N = 0\}$, and that by Fubini Theorem one has $|E^*| = |E|$.

We can now see that the Steiner symmetrization lowers the perimeter.

Lemma 8. *Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter, and let E^* be its Steiner symmetrized. Then*

$$P(E^*) \leq P(E). \tag{4}$$

Proof. First of all, we reduce ourselves to consider the case of a smooth set. Indeed, since smooth functions are dense in BV, we have a sequence $\{E_n\}$ of smooth sets whose characteristic functions strongly converge to χ_E in L^1, and with $P(E_n) \to P(E)$. Once the result has been proved for smooth sets, we have then

$$P(E) = \lim P(E_n) \geq \liminf P(E_n^*) \geq P(E^*),$$

where the last inequality is clear since by construction also the characteristic functions of E_n^* converge to χ_{E^*}. It is then enough to prove the inequality in the case of smooth sets.

We can further reduce ourselves to the case when E is a smooth set without vertical parts of the boundary, that is, the boundary of E is a finite union of graphs over closed subsets of $\mathbb{R}^{N-1}$. Indeed, this property is always true for a smooth set up to an arbitrarily small rotation.

Let us then consider such a set E. By definition, we can find finitely many disjoint open sets $A_j \subseteq \mathbb{R}^{N-1}$ such that

$$P(E) = \sum P(E, A_j \times \mathbb{R}) \tag{5}$$

and for each j there is some $K(j) \in \mathbb{N}$ and smooth functions $u^{\pm}_{i,j} : A_j \to \mathbb{R}$ for every $1 \leq i \leq K(j)$ in such a way that

$$u^-_{1,j} < u^+_{1,j} < u^-_{2,j} < u^+_{2,j} < \cdots < u^+_{K(j),j}$$

and that for every j one has

$$\partial E \cap (A_j \times \mathbb{R}) = \bigcup_{i=1}^{K(j)} \Big\{(y, t) : y \in A_j,\, u^-_{i,j} < t < u^+_{i,j}\Big\}.$$

Keep in mind that, if $A \subseteq \mathbb{R}^{N-1}$ is an open set and $u : A \to \mathbb{R}$ is a smooth function, then

$$\mathscr{H}^{N-1}\Big(\Big\{(y, u(y)),\, y \in A\Big\}\Big) = \int_A \sqrt{1 + |Du|^2(y)}\, d\mathscr{H}^{N-1}(y), \tag{6}$$

as one can immediately check.

Let us now observe that by construction we also have

$$P(E^*) = \sum P(E^*, A_j \times \mathbb{R}) \tag{7}$$

and by definition

$$E^* \cap (A_j \times \mathbb{R}) = \Big\{(y,t),\, y \in A_j,\, -\varphi(y) < t < \varphi(y)\Big\}\,,$$

where

$$\varphi(y) = \frac{\sum_{i=1}^{K(j)} u^+(y) - u^-(y)}{2}\,.$$

In particular,

$$|D\varphi(y)| = \left|\frac{\sum_{i=1}^{K(j)} Du^+_{i,j}(y) - Du^-_{i,j}(y)}{2}\right| \leq \frac{\sum_{i=1}^{K(j)} |Du^+_{i,j}(y)| + |Du^-_{i,j}(y)|}{2}\,.$$

Since the function $t \mapsto \sqrt{1+t^2}$ is increasing and strictly convex (hence also subadditive), we deduce that for every $y \in A_j$

$$2\sqrt{1+|D\varphi|^2(y)} \leq \sum_{i=1}^{K(j)} \sqrt{1+|Du^+_{i,j}|^2(y)} + \sqrt{1+|Du^-_{i,j}|^2(y)}\,.$$

Recalling then (6) we deduce

$$P(E^*, A_j \times \mathbb{R}) \leq P(E, A_j \times \mathbb{R})\,,$$

which by (5) and (7) concludes the thesis. □

Thanks to the result that we have just proved, if E is an isoperimetric set then so is also E^*. One would like to show the converse, that is, if E is isoperimetric then $E = E^*$ (up to a translation). However, this is not possible at this stage of the construction.

Suppose then that E is an isoperimetric set. As a consequence, the inequality $P(E^*) \leq P(E)$ is actually an equality. Let us give a quick look at the proof. Since the map $t \mapsto \sqrt{1+t^2}$ is *strictly* increasing and *strictly* convex, the only possibility for the inequality not to be strict is that $K(j) = 1$ for every j, and that $Du^+_{1,j} = -Du^-_{1,j}$. In particular, the inequality can be an equality only if all the sections are segments. However, notice carefully that this is true only for a smooth set E without vertical parts in the boundary! Indeed, a general set can be approximated by smooth sets without vertical parts of the boundary, but of course by approximation one cannot obtain a strict inequality. As a consequence, if E is a non-smooth set for which some sections are not segments, then we cannot say that $P(E^*) < P(E)$, and this is clearly unsatisfactory. To fix this problem, we need to prove a stronger version of (4) which implies the strict inequality for non-smooth sets with some sections which are not segments. This is precisely the content of the next result.

Lemma 9. *There exists a continuous, strictly increasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ such that the following holds. Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter, let E^* be its Steiner symmetrized, and let*

$$\Gamma = \Big\{x' \in \mathbb{R}^{N-1} :\, \mathscr{H}^0(\partial^* E_{x'}) > 2\Big\}\,. \tag{8}$$

Then,

$$P(E) \geq P(E^*) + P(E)\,\Psi\left(\frac{\mathscr{H}^{N-1}(\Gamma)}{P(E)}\right). \tag{9}$$

Before giving the proof, a comment is in order. Since the set E is a set of finite perimeter, it is defined up to a $\mathscr{H}^N$-negligible subset. In particular, the section $E_{x'}$ is defined up to a $\mathscr{H}^1$-negligible set for $\mathscr{H}^{N-1}$-a.e. x'. Thus, the set Γ is defined up to a $\mathscr{H}^{N-1}$-negligible subset, so the estimate of the lemma makes sense. Notice also that, for $\mathscr{H}^{N-1}$-a.e. x', the section $E_{x'}$ is a finite union of segments, thus $\mathscr{H}^0(\partial^* E_{x'})$ is an even number, and it is 0 if the section is empty, 2 if it is a segment, 4 if it is two segments and so on. In other words, $\mathscr{H}^0(\partial^* E_{x'}) > 2$ is the same as $\mathscr{H}^0(\partial^* E_{x'}) \geq 4$.

Proof. We start by noticing that it is sufficient to show (9) for smooth sets without vertical parts of the boundary. Indeed, given a generic set E with finite perimeter, again by the density of smooth functions in BV it is possible to find a sequence $\{E_n\}$ of smooth sets without vertical parts of the boundary such that $\chi_{E_n} \stackrel{*}{\rightharpoonup} \chi_E$ in BV, and moreover $P(E_n) \to P(E)$. It is simple to observe that, calling Γ_n the set given by (8) with E_n in place of E, one has

$$\mathscr{H}^{N-1}(\Gamma) \leq \liminf \mathscr{H}^{N-1}(\Gamma_n)$$

thanks to Vol'pert Theorem 16. Moreover, since $\chi_{E_n^*}$ converges to χ_{E^*} strongly in L^1, we also have

$$P(E^*) \leq \liminf P(E_n^*)\,.$$

As a consequence, since Ψ is increasing and continuous, if (9) is true for smooth sets without vertical parts of the boundary then we get

$$\begin{aligned}
P(E) = \lim_{n\to\infty} P(E_n) &\geq \liminf_{n\to\infty} P(E_n^*) + P(E_n)\Psi\left(\frac{\mathscr{H}^{N-1}(\Gamma_n)}{P(E_n)}\right) \\
&\geq \liminf_{n\to\infty} P(E_n^*) + \liminf_{n\to\infty} P(E_n)\Psi\left(\frac{\mathscr{H}^{N-1}(\Gamma_n)}{P(E_n)}\right) \\
&\geq P(E^*) + P(E)\liminf_{n\to\infty} \Psi\left(\frac{\mathscr{H}^{N-1}(\Gamma_n)}{P(E_n)}\right) = P(E^*) + P(E)\Psi\left(\liminf_{n\to\infty} \frac{\mathscr{H}^{N-1}(\Gamma_n)}{P(E_n)}\right) \\
&= P(E^*) + P(E)\Psi\left(\frac{\liminf_{n\to\infty} \mathscr{H}^{N-1}(\Gamma_n)}{P(E)}\right) \geq P(E^*) + P(E)\Psi\left(\frac{\mathscr{H}^{N-1}(\Gamma)}{P(E)}\right),
\end{aligned}$$

that is, (9) is true also for E. To conclude the proof we are then only reduced to show (9) for the case of a smooth set E without vertical parts of the boundary.

Let then E be such a set. Keeping in mind the proof of Lemma 8, and using the same notation, we have that

$$P(E) - P(E^*) \geq \int_\Gamma \tau(x')\,d\mathscr{H}^{N-1}(x')\,,$$

where for every $x' \in \Gamma$ the function τ is given by

$$\tau(x') = \left(\sum\nolimits_{i=1}^{K(j)} \sqrt{1 + |Du_{i,j}^+(x)|^2} + \sqrt{1 + |Du_{i,j}^-(x)|^2}\right) - 2\sqrt{1 + |D\varphi(x)|^2}\,,$$

being j the index such that $x' \in A_j$. Let us now fix $x' \in \Gamma$, and notice that by definition of Γ one has $K(j) \geq 2$. As a consequence, recalling again that $t \mapsto \sqrt{1+t^2}$ is strictly increasing and strictly convex, we get that

$$\tau(x') \geq 4\sqrt{1+\frac{|D\varphi(x')|^2}{4}} - 2\sqrt{1+|D\varphi(x')|^2}\,.$$

Writing for brevity

$$\Phi(t) = 4\sqrt{1+\frac{t^2}{4}} - 2\sqrt{1+t^2}\,,$$

we have then proved that

$$P(E) - P(E^*) \geq \int_\Gamma \Phi(|D\varphi(x')|)\, d\mathscr{H}^{N-1}(x')\,. \tag{10}$$

Notice that $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function with $\Phi(0) = 2$ and $\lim_{t\to\infty} \Phi(t) = 0$. Let us now divide Γ into two disjoint subsets Γ^- and Γ^+, both with measure $\mathscr{H}^{N-1}(\Gamma)/2$, and such that for every $y^- \in \Gamma^-$ and every $y^+ \in \Gamma^+$ one has $|D\varphi(y^-)| \leq |D\varphi(y^+)|$. Hence, there exists some $M \in \mathbb{R}^+$ such that $|D\varphi(x')| \leq M$ for every $x' \in \Gamma^-$, and $|D\varphi(x')| \geq M$ for every $x' \in \Gamma^+$. We have

$$P(E) \geq P(E^*) \geq \int_{\Gamma^+} 2\sqrt{1+|D\varphi(x')|^2}\, d\mathscr{H}^{N-1}(x') \geq \mathscr{H}^{N-1}(\Gamma)\sqrt{1+M^2} \geq \mathscr{H}^{N-1}(\Gamma) M\,,$$

from which we deduce

$$M \leq \frac{P(E)}{\mathscr{H}^{N-1}(\Gamma)}\,.$$

Using this inequality in (10), since Φ is decreasing we get

$$\begin{aligned} P(E) - P(E^*) &\geq \int_{\Gamma^-} \Phi(|D\varphi(x')|)\, d\mathscr{H}^{N-1}(x') \geq \mathscr{H}^{N-1}(\Gamma^-)\, \Phi(M) \\ &\geq \frac{\mathscr{H}^{N-1}(\Gamma)}{2}\, \Phi\left(\frac{P(E)}{\mathscr{H}^{N-1}(\Gamma)}\right). \end{aligned}$$

The proof is then concluded by defining

$$\Psi(t) = \frac{t}{2}\, \Phi(1/t)\,,$$

which is clearly a continuous and strictly increasing function with $\Psi(0) = 0$. □

Corollary 10. *If $E \subseteq \mathbb{R}^N$ is an isoperimetric set, then $\mathscr{H}^{N-1}(\Gamma) = 0$.*

As a consequence, we can then prove the convexity of isoperimetric sets. Notice that a set of finite perimeter is defined up to set of measure 0, hence we need to select a precise representative of E to speak about convexity. The best choice is to select the set of the Lebesgue points of E, that is, the set E^1 according with the notation of the Appendix.

Proposition 11. *Assume that $E \subseteq \mathbb{R}^N$ is an isoperimetric set. Then the set E^1 is convex.*

Proof. In the whole proof we will consider the representative $E = E^1$ for brevity of notation. Let x, y be two points in E, and let z be a point in the open segment xy. We have to show that $z \in E^1$. Up to a translation and rotation we can assume that $z = (0,0)$, $x = (0,-a)$ and $y = (0,b)$ with $a, b > 0$. Let us fix some $\varepsilon > 0$. By definition of points of density 1, there is some radius $\bar r = \bar r(\varepsilon) > 0$ such that for every $r < \bar r$

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} > 1 - \varepsilon\,, \qquad\qquad \frac{|E \cap B(y,r)|}{|B(y,r)|} > 1 - \varepsilon\,. \tag{11}$$

Let us consider any such r with $r \ll \min\{a, b\}$, and let us call $S = \{x' \in \mathbb{R}^{N-1} : |x'| \leq r/2\}$ the horizontal $(N-1)$-dimensional ball of radius $r/2$. We also call

$$D_x = \Big\{x' \in S : (x' \times \mathbb{R}) \cap E \cap B(x,r) = \emptyset\Big\}\,.$$

We have then

$$|B(x,r) \setminus E| \geq \mathscr{H}^{N-1}(D_x)\sqrt{3}r\,,$$

which by (11) implies

$$\mathscr{H}^{N-1}(D_x) \leq \frac{\varepsilon}{\sqrt{3}}\,\omega_N r^{N-1}\,.$$

The very same estimate clearly holds for the set D_y, defined replacing x by y. Calling then $S^- = S \setminus (\Gamma \cup D_x \cup D_y)$, and recalling that by Lemma 10 we have $\mathscr{H}^{N-1}(\Gamma) = 0$, we deduce

$$\mathscr{H}^{N-1}(S \setminus S^-) \leq \frac{2\varepsilon\omega_N}{\sqrt{3}}\,r^{N-1}\,.$$

Let us now consider a point $x' \in S^-$. Since $x' \notin D_x \cup D_y$, this means that there are $a(x') > a - r > r$ and $b(x') > b - r > r$ such that the two points $(x', -a(x'))$ and $(x', b(x'))$ belong to E, or in other words $-a(x')$ and $b(x')$ belong to $E_{x'}$. Since $x' \notin \Gamma$, the section $E_{x'}$ is a segment, which then must contain the whole segment $(-r, r)$. We have then obtained that $E \supseteq S^- \times (-r, r)$, hence

$$|B(z, r/2) \setminus E| \leq r\mathscr{H}^{N-1}(S \setminus S^-) \leq \frac{2\varepsilon\omega_N}{\sqrt{3}}\,r^N\,.$$

As a consequence,

$$\frac{|B(z,r/2) \cap E|}{|B(z,r/2)|} \geq 1 - \frac{2^{N+1}\varepsilon}{\sqrt{3}}\,.$$

Since this is true for every $r \ll 1$, we obtain that

$$\liminf_{r\to 0} \frac{|E \cap B(z,r)|}{|B(z,r)|} \geq 1 - \frac{2^{N+1}\varepsilon}{\sqrt{3}}\,,$$

and since $\varepsilon > 0$ was arbitrary we deduce that $z \in E^1$. The proof is then concluded. □

We can now start to show Theorem 1. We start with a particular case.

Proposition 12. *Let $E \subseteq \mathbb{R}^N$ be an isoperimetric set symmetric with respect to the origin, that is, $E = -E$. Then, E is a ball centered at the origin.*

Proof. Since E is isoperimetric, then E^1 is convex by Proposition 11. By the symmetry of E, we deduce that there exists a symmetric function $\ell : \mathbb{S}^{N-1} \to \mathbb{R}^+$ such that

$$E = \left\{\rho\theta : \theta \in \mathbb{S}^{N-1},\, 0 \leq \rho < \ell(\theta)\right\}.$$

To conclude the proof, we have then to prove that ℓ is constant (we actually know by symmetry that $\ell(\theta) = \ell(-\theta)$ for every direction $\theta \in \mathbb{S}^{N-1}$, but we will not use this fact).

Take two different directions $\theta \neq \nu \in \mathbb{S}^{N-1}$ with $\theta \cdot \nu \geq 0$, and let Π be the $(N-1)$-dimensional linear hyperplane containing ν and the $(N-2)$-dimensional space orthogonal to both θ and ν. Let us call P the point $P = \ell(\theta)\theta \in \partial E$, let Q be the point which is symmetric to P with respect to Π, and let $R = (P+Q)/2$. Notice that

$$R = \big(\ell(\theta)\theta \cdot \nu\big)\nu\,.$$

Since E is symmetric with respect to the origin, any linear hyperplane is bisecting E, then in particolar Π bisects E. We can then apply Lemma 4, and since E is isoperimetric we obtain that both the symmetrized sets E_1 and E_2 are isoperimetric. In particular, we call E_1 the one which coincides with E in the half-space containing P, so that both P and Q belong to $\partial^* E_1$.

Since E_1 is isoperimetric, then again by Proposition 11 we know that the set of its points of density 1 is convex. For every $0 < \lambda < 1$ both the points λP and λQ have density 1 on E_1, so λR has density 1 in E_1. Since R is in the direction ν, which belongs to Π, the fact that $\lambda R \in (E_1)^1$ implies $\ell(\nu) \geq |\lambda R| = \lambda\ell(\theta)\theta \cdot \nu$. Since this holds for every $0 < \lambda < 1$ we finally deduce

$$\ell(\nu) \geq \ell(\theta)\theta \cdot \nu\,.$$

This estimate is valid for every choice of $\nu \neq \theta \in \mathbb{S}^{N-1}$ with $\theta \cdot \nu \geq 0$. Calling $\alpha \in \mathbb{S}^1$ the angular distance between θ and ν, so that $\theta \cdot \nu = \cos(\alpha)$, the estimate can be rewritten as

$$\ell(\nu) \geq \ell(\theta)\cos(\alpha)\,. \tag{12}$$

Let now $M \in \mathbb{N}$ be a large number, to be eventually sent to ∞, and let us subdivide the angle between θ and ν in M equal angles. That is, we have $M+1$ directions $\theta_0 = \theta,\, \theta_1,\, \theta_2,\, \dots\,,\, \theta_M = \nu$ such that the angular distance between any θ_j and the corresponding θ_{j+1} is always α/M. The estimate (12) then implies

$$\ell(\nu) = \ell(\theta_M) \geq \ell(\theta_{M-1})\cos(\alpha/M) \geq \ell(\theta_{M-2})\cos(\alpha/M)^2 \geq \cdots \geq \ell(\theta)\cos(\alpha/M)^M\,.$$

Since, for any fixed α, $\lim_{M\to\infty}\cos(\alpha/M)^M = 1$, we finally deduce that $\ell(\nu) \geq \ell(\theta)$ or any two directions $\nu,\, \theta \in \mathbb{S}^{N-1}$ with $\theta \cdot \nu \geq 0$. From this, it obviously follows that ℓ is constant, that is, E is a ball. $\square$

It is now simple to conclude the proof of our main result in general.

Proof (of Theorem 1). Let $E \subseteq \mathbb{R}^N$ be an isoperimetric set of volume m, which exists by Lemma 3. Up to a translation, we can assume that

$$\Big|\Big\{x \in E : x_j > 0 \;\forall\, 1 \leq j \leq k\Big\}\Big| = \frac{m}{2^k} \qquad\qquad \forall\, 1 \leq k \leq N\,. \tag{13}$$

Lemma 4 implies that, whenever a hyperplane bisects an isoperimetric set, then both the corresponding symmetric sets (in the sense of Lemma 4) are also isoperimetric. Applying this argument a first time with the plane $\Pi_1 = \{x_1 = 0\}$, we obtain that $E_1 = \{x \in \mathbb{R}^N : (|x_1|, x_2, \dots, x_N) \in E\}$ is an isoperimetric set. We apply then the argument again to the set E_1 and the plane $\Pi_2 = \{x_2 = 0\}$. Notice that this is possible because by (13) we have that the plane Π_2 bisects E_1, while it is a priori not necessarily true that Π_2 bisects E. We obtain then that also $E_2 = \{x \in \mathbb{R}^N : (|x_1|, |x_2|, x_3 \dots, x_N) \in E\}$ is an isoperimetric set. Repeating $N-2$ times more the same argument we obtain that the set $E_N = \{x \in \mathbb{R}^N : (|x_1|, |x_2|, |x_3| \dots, |x_N|) \in E\}$ is an isoperimetric set. Since E_N is symmetric with respect to the origin, by Proposition 12 it is a ball. This means that (up to a translation) the intersection of E with the quadrant $\{x : x_j > 0 \; \forall 1 \leq j \leq N\}$ is a ball. Since we can repeat the same argument with any other quadrant, we immediately deduce that E is necessarily a ball.

Summarizing, we have proved that any isoperimetric set must be a ball. Since isoperimetric set exist, and since translations do not effect measure nor perimeter, the proof is concluded. □

Appendix. Some properties of sets of finite perimeter

To follow these notes a basic knowledge about sets of finite perimeter is needed. Given a set $E \subseteq \mathbb{R}^N$, one says that E is a set of locally finite perimeter if $\chi_E \in BV_{\rm loc}(\mathbb{R}^N)$. In this case, $P(E) = |D\chi_E|(\mathbb{R}^N)$. Only if the set is regular enough it is true that $P(E) = \mathscr{H}^{N-1}(\partial E)$, while in general the right term can be much bigger. Notice also that modifing the set E on a negligible set does not change the function χ_E in BV, so the perimeter does not change. The topological boundary ∂E, instead, can become much larger. For every Borel set $A \subseteq \mathbb{R}^N$, we will write $P(E, A) = |D\chi_E|(A)$. Let us generalize the notion of boundary.

Definition 13 (Reduced boundary)**.** *Let E be a set of finite perimeter, and let $x \in \mathbb{R}^N$. We say that x belongs to the* reduced boundary $\partial^* E$ *if there is some unit vector $\nu \in \mathbb{S}^{N-1}$ such that*

$$\lim_{r\to 0} \frac{|E \cap B(x,r) \cap \{(y-x)\cdot \nu < 0\}|}{|B(x,r)|} = \frac{1}{2}\,, \qquad \lim_{r\to 0} \frac{|E \cap B(x,r) \cap \{(y-x)\cdot \nu > 0\}|}{|B(x,r)|} = 0\,.$$

This vector $\nu = \nu_E(x)$ is called outer normal vector to E at x.

Definition 14 (Density and essential boundary)**.** *Let E be a set of finite perimeter, and let $x \in \mathbb{R}^N$. We say that the* density of E at x is $s \in [0,1]$ *if*

$$\lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = s\,,$$

and we call $E^s = \{x \in \mathbb{R}^N :$ the density of E at x is $s\}$. The essential boundary *is defined as $\partial_e E = \mathbb{R}^N \setminus (E^0 \cup E^1)$.*

We conclude by giving two results. The first one comes putting together fundamental contributions by Federer, Caccioppoli, De Giorgi, while the second one is due to Vol'pert.

Theorem 15. *For every set $E \subseteq \mathbb{R}^N$ of finite perimeter, one has*

$$P(E) = \mathscr{H}^{N-1}(\partial^* E)\,.$$

Moreover, the three sets $\partial^* E$, $\partial_e E$ *and* $E^{1/2}$ *coincide up to* $\mathscr{H}^{N-1}$*-negligible subsets.*

Theorem 16. *Let* $E \subseteq \mathbb{R}^N$ *be a set of finite perimeter. Then, for* $\mathscr{H}^{N-1}$*-a.e.* $x' \in \mathbb{R}^{N-1}$ *one has*

$$(\partial^* E)_{x'} = \partial^* (E_{x'})\,,$$

where the left term is the section of the boundary, that is, the set $\{t \in \mathbb{R} : (x', t) \in \partial^* E\}$*, and the right term is the boundary of the* 1*-dimensional section* $E_{x'}$.